

# TWO-PARAMETRIC DEFORMATION $U_{p,q}[gl(2/1)]$ AND ITS INDUCED REPRESENTATIONS

*In memory of Professor Asim Orhan Barut*

**Nguyen Anh Ky** <sup>a)</sup>

Theory Division, CERN, Geneve 23, CH-1211, **Switzerland**

## **Abstract**

The two-parametric quantum superalgebra  $U_{p,q}[gl(2/1)]$  is consistently defined. A construction procedure for induced representations of  $U_{p,q}[gl(2/1)]$  is described and allows us to construct explicitly all (typical and nontypical) finite-dimensional representations of this quantum superalgebra. In spite of some specific features, the present approach is similar to a previously developed method [1] which, as shown here, is applicable not only to the one-parametric quantum deformations but also to the multi-parametric ones.

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<sup>a)</sup> On leave of absence from the Institute of Physics, National Centre for Natural Science and Technology, P.O. Box 429, Bo Ho, Hanoi 10000, **Vietnam**

# I. Introduction

In [1], we suggested a method for explicit constructions of representations of the one-parametric quantum superalgebras  $U_q[gl(m/n)]$ . When applied to  $U_q[gl(2/2)]$ , this method allowed us to construct explicitly all (typical [1] and nontypical [2]) finite-dimensional representations of the latter quantum superalgebra. Certainly, as emphasized in Refs. 1 and 2, our method is also applicable for other quantum superalgebras and we could construct their representations in a similar way. Particularly, we can apply the method to, for example, multiparametric quantum superalgebras, [3–6], etc.. The multiparametric deformations were introduced [7] and since considered by a number of authors from different points of view (see, for example, Refs. 3–15). However, in spite of progresses in several aspects (e.g., group-space structures, differential calculus, exponential maps, etc.) representation theory is only well developed for a few simple cases like  $U_{p,q}[su(2)]$  (see for example Refs. 8),  $U_{p,q}[sl(2/1)]$ , [6], etc.. Here, in order to show once again the usefulness of the above-mentioned method we consider, as a further example, the two-parametric quantum superalgebra  $U_{p,q}[gl(2/1)]$  which, although resembles to the one-parametric quantum superalgebra  $U_{\sqrt{pq}}[gl(2/1)]$ , can not be identified with the latter. In this paper we suppose that both  $p$  and  $q$  are generic, i.e., not roots of unity. Following the approach of [1] we can directly construct explicit representations of the quantum superalgebra  $U_{p,q}[gl(2/1)]$  induced from some (usually, irreducible) finite-dimensional representations of the even subalgebra  $U_{p,q}[gl(2) \oplus gl(1)]$ . Since the latter is a stability subalgebra of  $U_{p,q}[gl(2/1)]$  we expect the constructed induced representations of  $U_{p,q}[gl(2/1)]$  are decomposed into finite-dimensional irreducible representations of  $U_{p,q}[gl(2) \oplus gl(1)]$ . For this purpose we shall introduce a  $U_{p,q}[gl(2/1)]$ -basis (i.e., a basis within a  $U_{p,q}[gl(2/1)]$ -module or briefly a basis of  $U_{p,q}[gl(2/1)]$ ) which will be convenient for us in investigating the module structure. This basis (see (3.10)) can be expressed in terms of some basis of the even subalgebra  $U_{p,q}[gl(2) \oplus gl(1)]$  which in turn represents a (tensor) product between a  $U_{p,q}[gl(2)]$ -basis and a  $gl(1)$ -factor. It will be shown that the finite-dimensional representations of  $U_{p,q}[gl(2)]$ , i.e., of  $U_{p,q}[gl(2) \oplus gl(1)]$  can be realized in the Gel'fand-Zetlin (GZ) basis. The finite-dimensional representations of  $U_{p,q}[gl(2/1)]$  constructed are irreducible and can be decomposed into finite-dimensional irreducible representations of the subalgebra  $U_{p,q}[gl(2) \oplus gl(1)]$ .

In section II we shall define the quantum superalgebra  $U_{p,q}[gl(2/1)]$  and consider how to construct its representations induced from representations of the subalgebra  $U_{p,q}[gl(2) \oplus gl(1)]$ . Finite-dimensional representations of  $U_{p,q}[gl(2/1)]$  are constructed in section III where the above-mentioned appropriate basis is described. The conclusion and some comments are given in section IV.

Throughout the paper we shall frequently use the following notation

$$[x] \equiv [x]_{p,q} := \frac{q^x - p^{-x}}{q - p^{-1}} \quad (1.1)$$

for quantum deformations of  $x$  which are operators or numbers,

$$[X, Y]_r := XY - rYX \quad (1.2)$$

for  $r$ -deformed commutators between two operators  $X$  and  $Y$  and

$$[m] \tag{1.3}$$

for the highest weights (signatures) of the Gel'fand–Zetlin basis vectors  $(m)$ . We hope this notation will not confuse the reader.

## II. $U_{p,q}[\mathfrak{gl}(2/1)]$ and its induced representations

The two-parametric quantum superalgebra  $U_{p,q}[\mathfrak{gl}(2/1)]$  is consistently defined through the generators  $E_{12}, E_{21}, E_{23}, E_{32}, E_{ii}, i = 1, 2, 3$ , and  $L$  satisfying

a) the super-commutation relations  $(1 \leq i, i+1, j, j+1 \leq 3)$ :

$$[E_{ii}, E_{jj}] = 0, \tag{2.1a}$$

$$[E_{ii}, E_{j,j+1}] = (\delta_{ij} - \delta_{i,j+1})E_{j,j+1}, \tag{2.1b}$$

$$[E_{ii}, E_{j+1,j}] = (\delta_{i,j+1} - \delta_{ij})E_{j+1,j}, \tag{2.1c}$$

$$[L, E_{12}] = [L, E_{21}] = [L, E_{ii}] = 0, \tag{2.1d}$$

$$[E_{12}, E_{21}] = \left(\frac{q}{p}\right)^{L-h_1/2} [h_1], \tag{2.1e}$$

$$\{E_{23}, E_{32}\} = \left(\frac{q}{p}\right)^{-h_2} [h_2]. \tag{2.1f}$$

$$h_i = (E_{ii} - \frac{d_{i+1}}{d_i} E_{i+1,i+1}), \tag{2.1g}$$

with  $d_1 = d_2 = -d_3 = 1$ ,

b) the Serre-relations:

$$\begin{aligned} E_{23}^2 &= E_{32}^2 = 0, \\ [E_{12}, E_{13}]_p &= [E_{21}, E_{31}]_q = 0, \end{aligned} \tag{2.2}$$

where

$$E_{13} := [E_{12}, E_{23}]_{q^{-1}},$$

and

$$E_{31} := -[E_{21}, E_{32}]_{p^{-1}}. \tag{2.3}$$

are defined as new odd generators which, as we can show, have vanishing squares. Now the extra-Serre relations are not necessary, unlike in higher rank cases [1,2,16]. The commutators between the maximal-spin operator  $L$  and the odd generators take concrete forms on concrete basis vectors.

These generators  $E_{ij}$ ,  $i, j = 1, 2, 3$ , are two-parametric deformation analogues of the Weyl generators  $e_{ij}$

$$(e_{ij})_{kl} = \delta_{ik}\delta_{jl}, \quad i, j, k, l = 1, 2, 3, \tag{2.4}$$

of the classical (i.e., non-deformed) superalgebra  $\mathfrak{gl}(2/1)$  whose universal enveloping algebra  $U[\mathfrak{gl}(2/1)]$  is a classical limit of  $U_{p,q}[\mathfrak{gl}(2/1)]$  when  $p, q \rightarrow 1$ .

From the relations (2.1)–(2.3) we see that every of the odd spaces  $A_{\pm}$

$$A_+ = \text{lin.env.}\{E_{13}, E_{23}\}, \quad (2.5)$$

$$A_- = \text{lin.env.}\{E_{31}, E_{32}\}, \quad (2.6)$$

is, as always, a representation space of the even subalgebra  $U_{p,q}[gl(2/1)_0] \equiv U_{p,q}[gl(2) \oplus gl(1)]$  which, generated by the generators  $E_{12}$ ,  $E_{21}$ ,  $L$  and  $E_{ii}$ ,  $i = 1, 2, 3$ , is a stability subalgebra of  $U_{p,q}[gl(2/1)]$ . Therefore, we can construct representations of  $U_{p,q}[gl(2/1)]$  induced from some (finite-dimensional irreducible) representations of  $U_{p,q}[gl(2/1)_0]$  which are realized in some representation spaces (modules)  $V_0^{p,q}$  being tensor products of  $U_{p,q}[gl(2)]$ -modules  $V_{0,gl_2}^{p,q}$  and  $gl(1)$ -modules (factors)  $V_{0,gl_1}^{p,q}$ . Following [1] we demand

$$E_{23}V_0^{p,q} = 0 \quad (2.7)$$

that is

$$U_{p,q}(A_+)V_0^{p,q} = 0. \quad (2.8)$$

In such a way we turn the  $U_{p,q}[gl(2/1)_0]$ -module  $V_0^{p,q}$  into a  $U_{p,q}(B)$ -module where

$$B = A_+ \oplus gl(2) \oplus gl(1). \quad (2.9)$$

The  $U_{p,q}[gl(2/1)]$ -module  $W^{p,q}$  induced from  $U_{p,q}[gl(2/1)_0]$ -module  $V_0^{p,q}$  is the factor-space

$$W^{p,q} = [U_{p,q} \otimes V_0^{p,q}] / I^{p,q} \quad (2.10)$$

where

$$U_{p,q} \equiv U_{p,q}[gl(2/1)], \quad (2.11)$$

while  $I^{p,q}$  is the subspace

$$I^{p,q} = \text{lin.env.}\{ub \otimes v - u \otimes bv \mid u \in U_{p,q}, b \in U_{p,q}(B) \subset U_{p,q}, v \in V_0^{p,q}\}. \quad (2.12)$$

Using the above-given commutation relations (2.1)–(2.2) and the definitions (2.3) we can prove the following analogue of the Poincaré–Birkhoff–Witt theorem

*Proposition 1:* The quantum deformation  $U_{p,q} := U_{p,q}[gl(2/1)]$  is spanned on all possible linear combinations of the elements

$$g = (E_{23})^{\eta_1} (E_{13})^{\eta_2} (E_{31})^{\theta_1} (E_{32})^{\theta_2} g_0, \quad (2.13)$$

where  $\eta_i, \theta_i = 0, 1$  and  $g_0 \in U_{p,q}[gl(2/1)_0] \equiv U_{p,q}[gl(2) \oplus gl(1)]$ .

Then we arrive at the next assertion

*Proposition 2:* The induced  $U_{p,q}[gl(2/1)]$ -module  $W^{p,q}$  is the linear span

$$W^{p,q}([m]) = \text{lin.env.}\{(E_{31})^{\theta_1} (E_{32})^{\theta_2} \otimes v \mid v \in V_0^{p,q}, \theta_1, \theta_2 = 0, 1\}, \quad (2.14)$$

which is decomposed into (four, at most) finite-dimensional irreducible modules  $V_k^{p,q}$  of the even subalgebra  $U_{p,q}[gl(2/1)_0]$

$$W^{p,q}([m]) = \bigoplus_{0 \leq k \leq 3} V_k^{p,q}([m]_k), \quad (2.15)$$

where  $[m]$  and  $[m]_k$  are some signatures (highest-weights) characterizing the module  $W^{p,q} \equiv W^{p,q}([m])$  and the modules  $V_k^{p,q} \equiv V_k^{p,q}([m]_k)$ , respectively.

As a consequence, for a basis in  $W^{p,q}$  we can take all the vectors of the form

$$|\theta_1, \theta_2; (m)\rangle := (E_{31})^{\theta_1} (E_{32})^{\theta_2} \otimes (m), \quad \theta_1, \theta_2 = 0, 1, \quad (2.16)$$

where  $(m)$  is a (GZ, for example,) basis in  $V_0^{p,q} \equiv V_0^{p,q}([m])$ . We refer to this basis as the induced  $U_{p,q}[gl(2/1)]$ -basis (or simply, the induced basis) in order to distinguish it from another  $U_{p,q}[gl(2/1)]$ -basis introduced later and called a reduced basis.

Any vector  $w$  from the module  $W^{p,q}$  can be represented as

$$w = u \otimes v, \quad u \in U_{p,q}, \quad v \in V_0^{p,q}. \quad (2.17)$$

Then  $W^{p,q}$  is a  $U_{p,q}[gl(2/1)]$ -module in the sense

$$gw \equiv g(u \otimes v) = gu \otimes v \in W^{p,q} \quad (2.18)$$

for  $g, u \in U_{p,q}$ ,  $w \in W^{p,q}$  and  $v \in V_0^{p,q}$ .

### III. Finite-dimensional representations of $U_{p,q}[gl(2/1)]$

We can show that finite-dimensional representations of  $U_{p,q}[gl(2/1)_0]$  can be realized in some spaces (modules)  $V_k^{p,q}$  spanned by the (tensor) products

$$\begin{bmatrix} m_{12} & m_{22} \\ m_{11} \end{bmatrix}; \begin{bmatrix} m_{32} = m_{31} \\ m_{31} \end{bmatrix} \equiv \begin{bmatrix} [m]_2 \\ m_{11} \end{bmatrix}; \begin{bmatrix} [m]_1 \\ m_{31} \end{bmatrix} \equiv (m)_{gl(2)} \otimes m_{31} \equiv (m)_k \quad (3.1a)$$

between the (GZ) basis vectors  $(m)_{gl(2)}$  of  $U_{p,q}[gl(2)]$  and the  $gl(1)$ -factors  $m_{31}$ , where  $m_{ij}$  are complex numbers such that

$$m_{12} - m_{11}, m_{11} - m_{22} \in \mathbf{Z}_+ \quad (3.1b)$$

and

$$m_{32} = m_{31}. \quad (3.1c)$$

Indeed, any finite-dimensional representation of (not only)  $U_{p,q}[gl(2)]$  is always highest weight and if the generators  $L$  and  $E_{ij}$ ,  $i, j = 1, 2$  are defined on (3.1) as follows

$$\begin{aligned} E_{11}(m)_k &= (l_{11} + 1)(m)_k, \\ E_{22}(m)_k &= (l_{12} + l_{22} - l_{11} + 2)(m)_k, \\ E_{12}(m)_k &= ([l_{12} - l_{11}][l_{11} - l_{22}])^{1/2} (m)_k^{+11}, \\ E_{21}(m)_k &= ([l_{12} - l_{11} + 1][l_{11} - l_{22} - 1])^{1/2} (m)_k^{-11}, \\ L(m)_k &= \frac{1}{2}(l_{12} - l_{22} - 1)(m)_k, \\ E_{33}(m)_k &= (l_{31} + 1)(m)_k, \end{aligned} \quad (3.2a)$$

$$l_{ij} = m_{ij} - (i - 2\delta_{i,3}), \quad (3.2b)$$

where a vector  $(m)_k^{\pm ij}$  is obtained from  $(m)$  by replacing  $m_{ij}$  with  $m_{ij} \pm 1$ , they really satisfy the commutation relations (2.1a)–(2.1e) for  $U_{p,q}[gl(2/1)_0]$ . The highest weight described by the first row (signature)

$$[m]_k = [m_{12}, m_{22}, m_{32}] \quad (3.3)$$

of the patterns (3.1) is nothing but an ordered set of the eigen-values of the Cartan generators  $E_{ii}$ ,  $i = 1, 2, 3$ , on the highest weight vector  $(M)_k$  defined as follows

$$E_{12}(M)_k = 0, \quad (3.4)$$

$$E_{ii}(M)_k = m_{i2}(M)_k, \quad (3.5)$$

The highest weight vector  $(M)_k$  can be obtained from  $(m)_k$  by setting  $m_{11} = m_{12}$

$$(M)_k = \left[ \begin{array}{cc} m_{12} & m_{22} \\ m_{12} & m_{31} \end{array} ; \begin{array}{c} m_{32} = m_{31} \\ m_{31} \end{array} \right]. \quad (3.6)$$

A lower weight vector  $(m)_k$  can be derived vice versa from  $(M)_k$  by the formula

$$(m)_k = \left( \frac{[m_{11} - m_{22}]!}{[m_{12} - m_{22}]![m_{12} - m_{11}]!} \right)^{1/2} (E_{21})^{m_{12} - m_{11}} (M)_k \quad (3.7)$$

In particular, for the case  $k = 0$ , instead of the above notations, we omit the subscript 0, i.e.,

$$(m)_0 \equiv (m); \quad [m]_0 \equiv [m]; \quad (M)_0 \equiv (M), \quad (3.8)$$

putting

$$m_{i2} = m_{i3}, \quad i = 1, 2, 3, \quad (3.9)$$

where  $m_{i3}$  are some of the complex values of  $m_{i2}$ , therefore,  $m_{13} - m_{11}$ ,  $m_{11} - m_{23} \in \mathbf{Z}_+$ . We emphasize that  $[m]$  and  $(M)$ , because of (2.7), are also, respectively, the highest weight and the highest weight vector in the  $U_{p,q}[gl(2/1)]$ -module  $W^{p,q} = W^{p,q}([m])$ . Characterizing the latter module as the whole,  $[m]$  and  $(M)$  are, respectively, referred to as the global highest weight and the global highest weight vector, while  $[m]_k$  and  $(M)_k$  are, respectively, the local highest weights and the local highest weight vectors characterizing only the submodules  $V^{p,q} = V^{p,q}([m]_k)$ .

Following the arguments of [1], for an alternative with (2.16) basis of  $W^{p,q}$  we can choose the union of all the bases (3.1) which are denoted now by the patterns

$$\left[ \begin{array}{ccc} m_{13} & m_{23} & m_{33} \\ m_{12} & m_{22} & m_{32} \\ m_{11} & 0 & m_{31} \end{array} \right]_k \equiv \left[ \begin{array}{cc} m_{12} & m_{22} \\ m_{11} & m_{31} \end{array} ; \begin{array}{c} m_{32} = m_{31} \\ m_{31} \end{array} \right]_k \equiv (m)_k, \quad (3.10)$$

where the first row  $[m] = [m_{13}, m_{23}, m_{33}]$  is simultaneously the highest weight of the submodule  $V^{p,q} = V^{p,q}([m])$  and the whole module  $W^{p,q} = W^{p,q}([m])$ , while the second row  $[m]_k = [m_{12}, m_{22}, m_{32}]$  is the local highest weight of some  $U_{p,q}[gl(2/1)_0]$ -module  $V_k^{p,q} = V_k^{p,q}([m]_k)$  containing the considered vector  $(m)_k$ . The basis (3.10) of  $W^{p,q}$  is called the  $U_{p,q}[gl(2/1)]$ -reduced basis or simply the reduced basis. The

latter, as mentioned before and shown later, is convenient for us in investigating the module structure of  $W^{p,q}$ .

Note once again that the condition

$$m_{32} = m_{31} \quad (3.1c)$$

has always to be fulfilled.

The highest weight vectors  $(M)_k$ , now, in the notation (3.10) have the form

$$(M)_k = \begin{bmatrix} m_{13} & m_{23} & m_{33} \\ m_{12} & m_{22} & m_{32} \\ m_{12} & 0 & m_{31} \end{bmatrix}_k \quad (3.11)$$

as for  $k = 0$  the notation given in (3.8) and (3.9) is also taken into account.

*Proposition 3:* The highest weight vectors  $(M)_k$  are expressed in terms of the induced basis (2.16) as follows

$$\begin{aligned} (M)_0 &= a_0 |0, 0; (M)\rangle, \quad a_0 \equiv 1, \\ (M)_1 &= a_1 |0, 1; (M)\rangle, \\ (M)_2 &= a_2 \left\{ |1, 0; (M)\rangle + q^{2l} [2l]^{-1/2} |0, 1; (M)^{-11}\rangle \right\}, \\ (M)_3 &= a_3 \{ |1, 1; (M)\rangle \}, \end{aligned} \quad (3.12a)$$

where  $a_i$ ,  $i = 0, 1, 2, 3$ , are some numbers depending, in general, on  $p$  and  $q$ , while  $l$  is

$$l = \frac{1}{2}(m_{13} - m_{23}) \quad (3.12b)$$

Indeed, all the vectors  $(M)_k$  given above satisfy the condition (3.4). From the formulae (3.5) and (3.12) the highest weights  $[m]_k$  can be easily identified

$$\begin{aligned} [m]_0 &= [m_{13}, m_{23}, m_{33}], \\ [m]_1 &= [m_{13}, m_{23} - 1, m_{33} + 1], \\ [m]_2 &= [m_{13} - 1, m_{23}, m_{33} + 1], \\ [m]_3 &= [m_{13}, m_{23}, m_{33} + 2] \end{aligned} \quad (3.13)$$

Using the rule (3.7) we obtain all the basis vectors  $(m)_k$

$$\begin{aligned} (m)_0 &\equiv \begin{bmatrix} m_{13} & m_{23} & m_{33} \\ m_{13} & m_{23} & m_{33} \\ m_{11} & 0 & m_{33} \end{bmatrix} = |0, 0; (m)\rangle, \\ (m)_1 &\equiv \begin{bmatrix} m_{13} & m_{23} & m_{33} \\ m_{13} & m_{23} - 1 & m_{33} + 1 \\ m_{11} & 0 & m_{33} + 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
&= a_1 \left\{ - \left( \frac{[l_{13} - l_{11}]}{[2l + 1]} \right)^{1/2} |1, 0; (m)^{+11}\rangle \right. \\
&\quad \left. + p^{l_{11} - l_{13}} \left( \frac{[l_{11} - l_{23}]}{[2l + 1]} \right)^{1/2} |0, 1; (m)\rangle \right\}, \\
(m)_2 &\equiv \begin{bmatrix} m_{13} & m_{23} & m_{33} \\ m_{13} - 1 & m_{23} & m_{33} + 1 \\ m_{11} & 0 & m_{33} + 1 \end{bmatrix} \\
&= a_2 \left\{ \left( \frac{q}{p} \right)^{l_{13} - l_{11} - 1} \left( \frac{[l_{11} - l_{23}]}{[2l]} \right)^{1/2} |1, 0; (m)^{+11}\rangle \right. \\
&\quad \left. + q^{l_{13} - l_{23} - 1} p^{l_{11} - l_{13} + 1} \left( \frac{[l_{13} - l_{11}]}{[2l]} \right)^{1/2} |0, 1; (m)\rangle \right\}, \\
(m)_3 &\equiv \begin{bmatrix} m_{13} & m_{23} & m_{33} \\ m_{13} - 1 & m_{23} - 1 & m_{33} + 2 \\ m_{11} & 0 & m_{33} + 2 \end{bmatrix} \\
&= a_3 |1, 1; (m)\rangle, \tag{3.14}
\end{aligned}$$

where  $l_{ij}$  and  $l$  are given in (3.2b) and (3.12b), respectively. Here, we omit the subscript  $k$  in the above patterns since there is no degeneration between them. The formulae (3.14), in fact, represent the way in which the reduced basis (3.10) is written in terms of the induced basis (2.16). From (3.14) we can derive their inverse relation

$$\begin{aligned}
|1, 0; (m)\rangle &= (m) \\
|1, 0; (m)\rangle &= -\frac{1}{a_1} q^{l_{11} - l_{23} - 1} \left( \frac{[l_{13} - l_{11} + 1]}{[2l + 1]} \right)^{1/2} (m)_1^{-11} \\
&\quad + \frac{1}{a_2 p} q^{l_{11} - l_{13}} \frac{([l_{11} - l_{23} - 1][2l])^{1/2}}{[2l + 1]} (m)_2^{-11}, \\
|0, 1; (m)\rangle &= \frac{1}{a_1} \left( \frac{[l_{11} - l_{23}]}{[2l + 1]} \right)^{1/2} (m)_1 \\
&\quad + \frac{1}{a_2} \left( \frac{p}{q} \right)^{l_{13} - l_{11} - 1} \frac{([l_{13} - l_{11}][2l])^{1/2}}{[2l + 1]} (m)_2, \\
|1, 1; (m)\rangle &= \frac{1}{c_3} (m)_3^{-11}. \tag{3.15}
\end{aligned}$$

Now we are ready to compute all the matrix elements of the generators in the basis (3.10). As we shall see, the latter basis allows a clear description of a decomposition of a  $U_{p,q}[gl(2/1)]$ -module  $W^{p,q}$  in irreducible  $U_{p,q}[gl(2/1)_0]$ -modules  $V_k^{p,q}$ . Since the



finite-dimensional representations of the  $U_{p,q}[gl(2/1)]$  in some basis are completely defined by the actions of the even generators and the odd Weyl–Chevalley ones  $E_{23}$  and  $E_{32}$  in the same basis, it is sufficient to write down the matrix elements of these generators only. For the even generators the matrix elements have already been given in (3.2), while for  $E_{23}$  and  $E_{32}$ , using the relations (2.1)–(2.3), (3.14) and (3.15) we have

$$\begin{aligned}
E_{23}(m) &= 0, \\
E_{23}(m)_1 &= a_1 \left(\frac{p}{q}\right)^{l_{23}+l_{33}+3} \left(\frac{[l_{11}-l_{23}]}{[2l+1]}\right)^{1/2} [l_{23}+l_{33}+3](m), \\
E_{23}(m)_2 &= a_2 \left(\frac{p}{q}\right)^{l_{23}+l_{33}+4} \left(\frac{[l_{13}-l_{11}]}{[2l]}\right)^{1/2} [l_{13}+l_{33}+3](m), \\
E_{23}(m)_3 &= a_3 \left(\frac{p}{q}\right)^{l_{13}+l_{23}+l_{33}-l_{11}+2} \left\{ \frac{1}{a_1 q} \left(\frac{[l_{13}-l_{11}]}{[2l+1]}\right)^{1/2} [l_{13}+l_{33}+3](m)_1, \right. \\
&\quad \left. - \frac{1}{a_2 p} ([l_{11}-l_{23}][2l])^{1/2} \frac{[l_{23}+l_{33}+3]}{[2l+1]}(m)_2 \right\} \tag{3.16a}
\end{aligned}$$

and

$$\begin{aligned}
E_{32}(m) &= \frac{1}{a_1} \left(\frac{[l_{11}-l_{23}]}{[2l+1]}\right)^{1/2} (m)_1, \\
&\quad + \frac{1}{a_2} \left(\frac{p}{q}\right)^{l_{13}-l_{11}-1} \frac{([l_{13}-l_{11}][2l])^{1/2}}{[2l+1]}(m)_2 \\
E_{32}(m)_1 &= \frac{a_1}{a_3} p \left(\frac{[l_{13}-l_{11}]}{[2l+1]}\right)^{1/2} (m)_3, \\
E_{32}(m)_{(2)} &= -\frac{a_2}{a_3} p \left(\frac{q}{p}\right)^{l_{13}-l_{11}-1} \left(\frac{[l_{11}-l_{23}]}{[2l]}\right)^{1/2} (m)_3, \\
E_{32}(m)_3 &= 0. \tag{3.16b}
\end{aligned}$$

*Proposition 4:* The finite-dimensional representations (3.16) of  $U_{p,q}[gl(2/1)]$  are irreducible and called typical if only if the condition

$$[l_{13}+l_{33}+3][l_{23}+l_{33}+3] \neq 0 \tag{3.17}$$

holds.

When this condition (3.17) is violated, i.e. one of the following condition pairs

$$[l_{13}+l_{33}+3] = 0 \tag{3.18a}$$

and

$$[l_{23} + l_{33} + 3] \neq 0 \quad (3.18b)$$

or

$$[l_{13} + l_{33} + 3] \neq 0 \quad (3.19a)$$

and

$$[l_{23} + l_{33} + 3] = 0 \quad (3.19b)$$

(but not both (3.18a) and (3.19b) simultaneously) holds, the module  $W^{p,q}$  is no longer irreducible but indecomposable. However, there exists an invariant subspace, say  $I_k^{p,q}$ , of  $W^{p,q}$  such that the factor-representation in the factor-module

$$W_k^{p,q} := W^{p,q} / I_k^{p,q} \quad (3.20)$$

is irreducible. We say that is a nontypical representation in a nontypical module  $W_k^{p,q}$ . Then, as in [2], it is not difficult for us to prove the following assertions

*Proposition 5:*

$$V_3^{p,q} \subset I_k^{p,q}, \quad (3.21)$$

and

$$V_0^{p,q} \cap I_k^{p,q} = \emptyset. \quad (3.22)$$

From (3.16)–(3.18) we can easily find all nontypical representations of  $U_{p,q}[gl(2/1)]$  which are classified into two classes.

### III.1. Class 1 nontypical representations:

This class is characterized by the conditions (3.18a) and (3.18b) which for generic  $p$  and  $q$  take the forms

$$l_{13} + l_{33} + 3 = 0, \quad (3.18a')$$

and

$$l_{23} + l_{33} + 3 \neq 0, \quad (3.18b')$$

respectively. In other words, we have to replace everywhere all  $m_{33}$  by  $-m_{13} - 1$  and keep (3.18b') valid. Thus we have

*Proposition 6:*

$$I_1^{p,q} = V_3^{p,q} \oplus V_2^{p,q}. \quad (3.23)$$

Then the class 1 nontypical representations in

$$W_1^{p,q} = W_1^{p,q}([m_{13}, m_{23}, -m_{13} - 1]) \quad (3.24)$$

are given through (3.16) by keeping the conditions (3.18) (i.e., (3.18a') and (3.18b')) and replacing all vectors belonging to  $I_1^{p,q}$  with 0:

$$E_{23}(m) = 0,$$

$$E_{23}(m)_1 = a_1 \left( \frac{p}{q} \right)^{l_{23} - l_{13}} \left( \frac{[l_{11} - l_{23}]}{[2l + 1]} \right)^{1/2} [l_{23} - l_{13}](m) \quad (3.25a)$$

and

$$\begin{aligned} E_{32}(m) &= \frac{1}{a_1} \left( \frac{[l_{11} - l_{23}]}{[2l + 1]} \right)^{1/2} (m)_1 \\ E_{32}(m)_1 &= 0. \end{aligned} \quad (3.25b)$$

### III.2. Class 2 nontypical representations:

For this class nontypical representations we must keep the conditions

$$l_{13} + l_{33} + 3 \neq 0, \quad (3.19a')$$

and

$$l_{23} + l_{33} + 3 = 0. \quad (3.19b')$$

derived respectively from (3.19a) and (3.19b) when the deformation parameters  $p$  and  $q$  are generic. Equivalently, we have to replace everywhere all  $m_{33}$  by  $-m_{23}$  and keep (3.19a') valid.

Now the invariant subspace  $I_2^{p,q}$  is the following

*Propositions 7:*

$$I_2^{p,q} = V_3^{p,q} \oplus V_1^{p,q}. \quad (3.26)$$

The class 2 nontypical representations in

$$W_2^{p,q} = W_2^{p,q}([m_{13}, m_{23}, -m_{23}]) \quad (3.27)$$

are also given through (3.16) but by keeping the conditions (3.19) (i.e., (3.19a') and (3.19b')) valid and replacing all vectors belonging to the invariant by 0 subspace  $I_2^{p,q}$ :

$$\begin{aligned} E_{23}(m) &= 0, \\ E_{23}(m)_2 &= a_1 \frac{p}{q} \left( \frac{[l_{13} - l_{11}]}{[2l]} \right)^{1/2} [2l + 1](m) \end{aligned} \quad (3.28a)$$

and

$$\begin{aligned} E_{32}(m) &= \frac{1}{a_2} \frac{([l_{13} - l_{11}][2l])^{1/2}}{[2l + 1]} (m)_2 \\ E_{32}(m)_2 &= 0. \end{aligned} \quad (3.28b)$$

In order to complete this section we emphasize that nontypical representations have only been well investigated for a few cases of both classical and quantum superalgebras (see, in this context, the Conclusion in Ref. 2 and also some comments in Ref. 17). Therefore, the present results can be considered as a small step forward to this direction.

## IV. Conclusion

We have just defined the two-parametric quantum superalgebra  $U_{p,q}[gl(2/1)]$  and constructed at generic deformation parameters all its typical and nontypical representations leaving the coefficients  $a_i$ ,  $i = 1, 2, 3$ , as free parameters which can be fixed by some additional conditions, for example, the hermiticity condition. As an intermediate step (which, however, is of independent interest) we also introduced the reduced basis (3.10) which, as it is an extension of the Gel'fand–Zetlin basis to the present case, is appropriate for a clear description of decompositions of  $U_{p,q}[gl(2/1)]$ -modules into irreducible  $U_{p,q}[gl(2/1)_0]$ -modules. Although the present approach has some specific features it is similar to the one in Ref. 1. That shows once again the usefulness of the method of Ref. 1 which is thus applicable not only to the one-parametric quantum deformations but also to the multi-parametric ones.

As the general procedure has been given, the next step is to consider the case of non-generic  $p$  and  $q$  or to construct representations of larger quantum superalgebras like  $U_{p,q}[gl(n/1)]$ ,  $U_{p,q}[gl(n/m)]$ , etc. for both generic and non-generic deformation parameters. Let us emphasize once again that our approach avoids the use of the Clebsch–Gordan coefficients which are not always known, especially for higher rank (classical and quantum) algebras and multi-parametric deformations.

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